Maths for Computing Tutorial 4

1. Find a winning strategy for the 1st player in the game of Chomp when cookies are $2 \times n$, that is, 2 rows and *n* columns.

Solution: 1st player on her first turn picks the bottom-right cookie. Now, after every pair of turns of 2nd player and 1st player, 1st player will ensure that the layout will look like $2 \times l$ with a bottom-right cookie missing. Do you see how it can be done? In the end, it will come down to just poisoned cookie, if player 2 has not already eaten it.

2. Find a winning strategy for the 1st player in the game of Chomp when cookies are $n \times n$. **Solution:** 1st player picks (2,2)th cookie (diagonally next to the poisoned cookie). Now the layout will look like below:



After that 1st player will start mimicking the 2nd player. If player 2 picks kth cookie from the column then player 1 will pick the kth cookie from the row and vice-versa. Finally, only the poisoned cookie will be left and 2nd player will have to eat it.

3. Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer *n*.

Solution: Basis Step: It's trivially true as for n = 0, $7^{n+2} + 8^{2n+1} = 57$. **Inductive Step:** Assume P(k) is true, i.e., $7^{k+2} + 8^{2k+1}$ is divisible by 57. In order to prove P(k + 1) we need to show that $7^{k+3} + 8^{2k+3}$ is divisible by 57. Now,

$$7^{k+3} + 8^{2k+3} = 7.7^{k+2} + 64.8^{2k+1}$$

= 7.7^{k+2} + (57 + 7).8^{2k+1}
= 7.7^{k+2} + 7.8^{2k+1} + 57.8^{2k+1}
= 7(7^{k+2} + 8^{2k+1}) + 57.8^{2k+1}

Both $7(7^{k+2} + 8^{2k+1})$ and 57.8^{2k+1} are divisible by 57. Hence, $7^{k+3} + 8^{2k+3}$ is also divisible by 57.

4. Let $a_1 = 5$, and let $a_{n+1} = a_n^2$. Prove that the last *n* digits of a_n are the same as the last *n* digits of a_{n+1} .

Solution: It is sufficient to prove that $a_{n+1} - a_n$ is divisible by 10^n . We will do it by induction. **Basis Step:** Trivially true as $a_2 - a_1 = 20$ is divisible by 10^1 .

Inductive Step: Assume $a_{k+1} - a_k$ is divisible by 10^k . We need to prove that $a_{k+2} - a_{k+1}$ is divisible by 10^{k+1} .

Now, $a_{k+2} - a_{k+1} = a_{k+1}^2 - a_k^2 = (a_{k+1} + a_k)(a_{k+1} - a_k)$. Since $a_{k+1} - a_k$ is divisible by 10^k and $a_{k+1} + a_k$ is divisible by 10 as all a_i s end with 5, we can say that $(a_{k+1} + a_k)(a_{k+1} - a_k)$ is divisible by 10^{k+1} .

5. Show that it is possible to arrange the numbers 1, 2, ..., n in a row so that average of any two of these numbers never appear between them. [*Hint:* Show that it suffices to prove this fact when n is a power of 2. Then use mathematical induction to prove the result when is a power of 2.] **Solution:** It is sufficient to prove the statement for $n = 2^k$. Because if n is not a power of 2, we can consider the numbers $1, 2, 3, ..., n, ..., 2^k$ where 2^k is the smallest power of 2 that is larger than n. We will first arrange $1, 2, 3, ..., n, ..., 2^k$ in the desired order and then delete the numbers larger than n. Since averages of any two numbers were not present before deleting, they cannot appear after deleting as well.

Will now prove the statement for $1, 2, ..., 2^n$ using induction.

Basis Step: For n = 1 the numbers can simply be arranged as (1,2).

Inductive Step: Assume that we can arrange $1, 2, ..., 2^k$ numbers in the desired order as $a_1, a_2, ..., a_{2^k}$. Then the desired order for $1, 2, ..., 2^{k+1}$ will be $2a_1 - 1, 2a_2 - 1, ..., 2a_{2^k} - 1$, $2a_1, 2a_2, ..., 2a_{2^k}$. Cleary all the number from 1 to 2^{k+1} are present in this sequence. We will show now that no two numbers have their average between them. Suppose not, then these numbers will be of type $(2a_i - 1, 2a_j), (2a_i - 1, 2a_j - 1), (2a_i, 2a_j). (2a_i - 1, 2a_j)$ is not possible as the average will be not be an integer. $(2a_i, 2a_j)$ is also not possible because if some number $2a_k$ between $2a_i$ and $2a_j$ is the average of $2a_i$ and $2a_j$, then $2a_k = (2a_i + 2a_j)/2$. This further implies that a_k is the average of a_i and a_j , which is a contradiction as from IH no two numbers a_i and a_j have their average between them in $a_1, a_2, ..., a_{2^k}$. Similarly, $(2a_i - 1, 2a_j - 1)$ is also not possible.

6. In a party there are 2n participants, where *n* is a positive integer. Some participants shake hands with other participants. It is known that there do not exist three participants who have shaken hands with each other. Prove that the total number of handshakes is not more than n^2 . **Solution:** We will prove it using induction.

Basis Step: For n = 1, there will be just one $< 1^2$ handshake.

Inductive Step: Assume the statement is true for 2k participants. We will prove now that for 2(k + 1) = 2k + 2 participants handshakes can be at most $(k + 1)^2$. Among them pick any two participants, say p_1 and p_2 , who have shaken hands with each other. If we cannot find such two participants, then we are trivially done.

In the rest of the 2k players, there cannot be a player who has shaken hands with both p_1 and p_2 . Otherwise, it will violate the condition of no three participants shaking hands with each other. So the number of handshakes 2k players have with p_1 and p_2 can be at most 2k, the number of handshakes within 2k players is at most k^2 (from IH), and the number of handshakes between p_1 and p_2 is 1. If we sum all these numbers it will be at most $2k + k^2 + 1 = (k + 1)^2$.

7. At a tennis tournament, there were 2^n participants, where *n* is a positive integer, and any two of them played against each other exactly one time. Prove that we can find n + 1 players that can form a line in which everybody has defeated all the players who are behind him in the line. [*Hint:* Use induction and winner.]

Solution: Basis Step: For n = 1, there will be just 1 match and we can line up the players as (loser, winner).

Inductive case: Assume that 2^k players can be lined in the desired order. Let's see the tournament of 2^{k+1} players. We claim that winner of the tournament must have won at least 2^k matches. If not, then every player has won at most $2^k - 1$ matches. There are 2^{k+1} players, so total number of matches will be at most 2^{k+1} . $(2^k - 1)$ (we can count the total number of matches by adding the number of matches won by every player). But 2^{k+1} . $(2^k - 1)$ is less than (2^{k+1})

 $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$, the actual number of matches in a tournament of 2^{k+1} players, which is a contradiction.

Let's consider the matches played by any 2^k players that were defeated by the winner. We can call all those matches a tournament of 2^k players and by inductive hypothesis we can find k + 1 players in the desired order. Now we only need to add the winner in the front.